Further Maths Pure Notes

Complex Numbers

An imaginary number is a number of the form bi $(i = \sqrt{-1})$.
A complex number is written in the form $a + bi$. Complex numbers can be added or subtracted by adding/subtracting their real and imaginary parts.
If $b^2 - 4ac < 0$ for $ax^2 + bx + c = 0$, then it has two distinct complex roots, which are a complex conjugate pair.

$z = a + bi$ has a complex conjugate $z^* = a - bi$

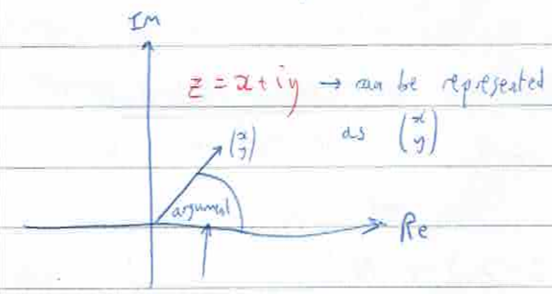
for $\underset{\text{real coefficients}}{ax^2 + bx + c = 0}$, if $b^2 - 4ac < 0$ then the roots are two complex conjugate numbers. Obvious but easy to forget $(x - \alpha)(x - \beta) = ax^2 + bx + c$ (α, β roots)

$\alpha = \beta^*$

for $aw^3 + bw^2 + cw + d = 0$, either all 3 roots are real, or one root is real and 2 are a complex conjugate pair.
for $az^4 + bz^3 + cz^2 + dz + e = 0$, either all 4 roots are real, 2 are real and two are a complex conjugate pair, or the 4 roots are 2 complex conjugate pairs.

Argand diagrams

You can represent complex numbers on an Argand diagram, the x-axis on an Argand diagram is the real axis and the y-axis is the imaginary axis.



$|z| = \sqrt{x^2 + y^2}$ $\tan(\theta) = \frac{y}{x}$

might have to include π somewhere here if in other quadrant

$-\pi < \theta \leq \pi$

For a complex number $z = a + bi$, where $|z| = r$ and $\arg z = \theta$
the modulus-argument form of z is $z = r(\cos\theta + i\sin\theta)$

for any two complex numbers z_1 and z_2:

$|z_1 z_2| = |z_1||z_2|$ $\qquad$ $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$\dfrac{|z_1|}{|z_2|} = \left|\dfrac{z_1}{z_2}\right|$ $\qquad$ $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

$|z_2 - z_1|$ = distance between the points z_1 and z_2

Circle: Given $z_1 = x_1 + iy_1$, a circle with centre $= (x_1, y_1)$ is

$$\left.\begin{aligned} |z - z_1| &= r \\ |z - (x_1 + iy_1)| &= r \\ |z - x_1 - iy_1| &= r \end{aligned}\right\} \text{locus of points } z \text{ is a circle}$$

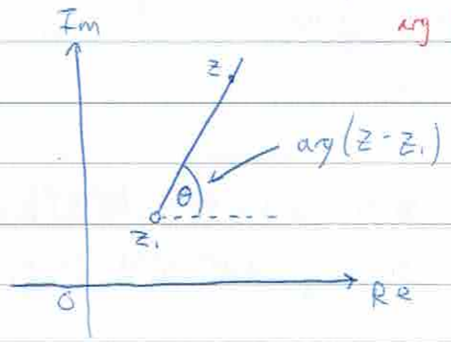
Perpendicular bisector: Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the locus of the points z is the perpendicular bisector of the line segment joining z_1 and z_2 where:

$$|z - z_1| = |z - z_2|$$

↳ distance from z_1 to z = distance from z_2 to z

Half line: Given $z_1 = x_1 + iy_1$, the locus of points z is a half line from, but not including, the fixed point z_1 making an angle θ with a line from the fixed point z_1 parallel to the real axis where:

$$\arg(z - z_1) = \theta$$



Further Complex numbers

Euler's relation, $e^{i\theta} = \cos\theta + i\sin\theta$

Any complex number can be written $z = r(\cos\theta + i\sin\theta)$ = mod-arg form, so:

$$z = re^{i\theta} \qquad r = |z|, \quad \theta = \arg z$$

if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$$

De Moivre's Theorem: $\left(r(\cos\theta + i\sin\theta)\right)^n = r^n(\cos n\theta + i\sin n\theta)$

$$2\cos\theta = z + \frac{1}{z} \qquad 2\cos n\theta = z^n + \frac{1}{z^n}$$

$$2i\sin\theta = z - \frac{1}{z} \qquad 2i\sin n\theta = z^n - \frac{1}{z^n}$$

$w, z \in \mathbb{C}$

$$\sum_{r=0}^{n-1} wz^r = w + wz + wz^2 + \dots + wz^{n-1} = \frac{w(z^n - 1)}{z - 1}$$

$$\sum_{r=0}^{\infty} wz^r = w + wz + wz^2 + \dots = \frac{w}{1-z}, \quad |z| < 1$$

if w and z are complex numbers (non-zero also) and n is a positive integer, then the equation $z^n = w$ has n roots

$$z = r(\cos\theta + i\sin\theta) = r\left(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\right)$$

hard part

In general, the solutions to $z^n = 1$ are $z = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) = e^{\frac{2\pi i k}{n}}$

these are known as the n^{th} roots of unity (roots of unity all have modulus = 1)

If n is a positive integer, then there is an n^{th} root of unity $\omega = e^{\frac{2\pi i}{n}}$ such that:

- The n^{th} roots of unity are $1, \omega, \omega^2, \dots, \omega^{n-1}$
- $1, \omega, \omega^2, \dots, \omega^{n-1}$ form the vertcies of a regular n-gon
- $1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0$

The n^{th} roots of any complex number s lie on the vertices of a regular n-gon with its centre at the origin.

If z_1 is one root of the equation $z^n = s$, and $1, \omega, \omega^2, \dots, \omega^{n-1}$ are the n^{th} roots of unity, then the roots of $z^n = s$ are:

$$z_1, z_1\omega, z_1\omega^2, \dots, z_1\omega^{n-1}$$

Series

Sum of a series of constant terms: $\sum_{r=1}^{n} 1 = n$

Formula for the sum of the first n natural numbers: $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Sum of a series that does not start at $r=1$: $\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$

Rearrange: $\sum_{r=1}^{n} k f(r) = k \sum_{r=1}^{n} f(r)$ $\sum_{r=1}^{n} \left(f(r) + g(r)\right) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$

Sum of squares of natural numbers is: $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

Sum of cubes of the natural numbers: $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Roots of polynomials

Root notation: the roots of an equation are written $\alpha, \beta, \gamma, \delta$

$ax^2 + bx + c = 0$	$ax^3 + bx^2 + cx + d = 0$	$ax^4 + bx^3 + cx^2 + dx + e = 0$
$\alpha + \beta = -\frac{b}{a}$	$\alpha + \beta + \gamma = -\frac{b}{a}$	$\alpha + \beta + \gamma + \delta = -\frac{b}{a}$
$\alpha\beta = \frac{c}{a}$	$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$	$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$
	$\alpha\beta\gamma = -\frac{d}{a}$	$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$
		$\alpha\beta\gamma\delta = \frac{e}{a}$

in general:

singles: $\alpha + \beta + \gamma \dots = \frac{-b}{a}$

pairs: $\alpha\beta + \alpha\gamma + \dots = \frac{c}{a}$

triples: $\alpha\beta\gamma + \alpha\beta\delta + \dots = -\frac{d}{a}$

quads: $\alpha\beta\gamma\delta = \frac{e}{a}$

$\alpha^n \times \beta^n \times \gamma^n \dots = (\alpha\beta\gamma\dots)^n$

= [singles^2 − 2(pairs)]

Sum of squares: $\alpha^2 + \beta^2 + \dots + = (\alpha + \beta + \dots)^2 - 2(\alpha\beta + \dots$

Sum of cubes: $\alpha^3 + \beta^3 + \dots + = (\alpha + \beta + \dots)^3 - 3(\alpha + \beta \dots)(\alpha\beta \dots$

$+ 3(\alpha\beta\gamma \dots)$

= singles^2 − 3(singles × pairs) + 3 tr

if "$g(x)$ ... has roots α, β and γ ..."

"... find equation with roots $f(\alpha), f(\beta), f(\gamma)$"

just do $w = f(x) \Rightarrow x = f^{-1}(w)$

then sub x into original equation

Volumes of Revolution

if $y = f(x)$ is rotated about the x-axis between $x = a$, and $x = b$ through 2π radians

$$\text{Volume} = \pi \int_a^b y^2 \cdot dx$$

if $y = f(x)$ is rotated about the y-axis between $y = a$ and $y = b$ through 2π radi

$$\text{Volume} = \pi \int_a^b x^2 \cdot dy$$

Cylinder volume: $V = \pi r^2 h$

Cone volume: $V = \frac{1}{3}\pi r^2 h$

If $x = f(t)$, $y = g(t)$ is rotated 2π between $x = a$ and $x = b$ about x-axis

$$\text{Volume} = \pi \int_a^b y^2 \cdot dx = \pi \int_q^p y^2 \frac{dx}{dt} \cdot dt$$

If $x = f(t)$, $y = g(t)$ is rotated 2π between $y = a$ and $y = b$ about y-axis

$$\text{Volume} = \pi \int_a^b x^2 \cdot dy = \pi \int_q^p x^2 \frac{dy}{dt} \cdot dt$$

don't forget to change the limits

Proof by Induction

You can use proof by induction to prove that a general statement is true for all positive integers.

1. Basis: prove that the statement is true for $n = 1$ [$p(1)$ is true]
2. Assumption: Assume that the statement is true for $n = k$ [assume $p(k)$ is true]
3. Inductive: Show that the general statement is true for $n = k+1$ [$p(k+1)$ is true given p
4. Conclusion: Therefore the statement is true for all positive integers, $n \in \mathbb{N}$

Matrices

Square matrix: a matrix where the number of rows and columns are the same

Zero matrix: a matrix where all of the numbers are 0 (denoted by 0)

Identity matrix: a square matrix in which all the numbers in the leading diagonal are 1 and the rest are 0 (denoted by I_k, where k describes the size)

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

size can be denoted by no. rows × no. columns eg $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$ has size 3×2

Matrices cont.

- To add or subtract matrices, you add or subtract the corresponding elements in each matrix (you can only do this with matrices of the same size eg. $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$)
- To multiply by a scalar, you multiply every element by that scalar eg $3\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \end{pmatrix}$
- Matrices can be multiplied together if the number of columns in the first is equal to the number of rows in the second matrix (here they are multiplicatively conformable)
- To multiply two matrices you multiply the elements in the left-hand matrix rows by the corresponding elements in each column in the right-hand matrix, then add the results

Determinants

eg. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ae+bf \\ ce+df \end{pmatrix}$

Note. $\det(M)$ can be written $|M|$

For a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(M) = ad - bc$

For a 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $\det(A) = a[\text{minor}(a)] - b[\text{minor}(b)] + c[\text{minor}(c)]$

If $\det A = 0$, A is singular. If $\det A \neq 0$, A is non-singular.

The minor of an element in a 3×3 matrix is the determinant of the 2×2 matrix left over when that element's row and column are removed

Inversion

if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

if $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then $A^{-1} = \frac{1}{\det A}(C_{\text{minor}})^T \Rightarrow A^{-1} = \frac{1}{\det A}(C_{\text{minor}})^T$

$MM^{-1} = M^{-1}M = I$

$(C_{\text{minor}})^T$: construct a matrix of minors (replace each element with its minor)

Form a matrix of cofactors (multiply the matrix of minors by alternating + and − signs)

Transpose this matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

If A and B are non-singular $(AB)^{-1} = B^{-1}A^{-1}$

If $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = V$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}V$

Plane consistency

A system of linear equations is consistent if there is at least one set of values that satisfies all simultaneously

If $\det A \neq 0$, then they are consistent and have 1 solution : planes meet at a point

if $\det A = 0$:
- Sheaf : equations are consistent with infinite solutions ← all equations are not equivalent
- All same plane : equations are consistent with infinite solutions ← all equations equivalent
- Prism : equations are inconsistent, and have no solutions
- 2+ parallel and not same : equation are inconsistent and have no solutions

eliminate a variable

To determine consistency find ④ and ⑤ from the three equations. If there is a set(s) of solutions which fulfill both ④ and ⑤ they are consistent, if not, they are inconsistent.

Matrix to be tested

- non-singular ($\det A \neq 0$) → consistent with one solution ⇨ meet at point
- Singular ($\det A = 0$)
  - consistent → equations are:
    - equivalent ⇨ same plane
    - not equivalent ⇨ sheaf
  - inconsistent → 2+ of the equations are:
    - parallel ⇨ 2+ planes parallel and non-identical
    - not parallel ⇨ prism

↳ basically: singular ⇒ consistent → parallel

Linear Transformations

Linear transformations always map the origin onto itself. Any linear transformation can be represented by a matrix

The linear transformation $T: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$ can be represented by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$

2-dimensions:

- Reflection in y-axis $= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ — Points on y-axis = invariant points; $x=0$, $y=k$ = invariant lines
- Reflection in x-axis $= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ — Points on x-axis = invariant points; $y=0$, $x=k$ = invariant lines
- Reflection in $y=x$ $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ — Points on $y=x$ = invariant points; $y=x$, $y=-x+k$ = invariant lines
- Reflection in $y=-x$ $= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ — Points on $y=-x$ = invariant points; $y=-x$, $y=x+k$ = invariant lines
- Rotation through angle θ anticlockwise $= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ — $(0,0)$ = invariant point; if $\theta = 180°$ = invariant lines
- Stretch of scale factor a parallel to x-axis and scale factor b parallel to the y-axis $= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ if $a=b$ ⇒ enlargement scale factor a : figure out invariant points and lines

For any linear transformation M: $\det M$ = scale factor for change in area (area scale factor)

3-dimensions:

- Reflection in plane $x=0$: $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ — Rotation θ anticlockwise about the x-axis: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ — A^{-1} reverses A
- Reflection in plane $y=0$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ — Rotation θ anticlockwise about the y-axis: $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$
- Reflection in plane $z=0$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ — Rotation θ anticlockwise about the z-axis: $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

A = point (can be a general point on a line)
AB = a line
$\vec{AB}$ = vector from A to B
$\underline{a}$ = position vector of A
$\hat{a}$ = unit vector of $\underline{a}$ $\quad \hat{a} = \frac{\underline{a}}{|\underline{a}|}$

This is all 3D btw

Vectors

Vector equation of a line:

Position vector of a general point; λ is a scalar parameter

Passes through A (position vector $\underline{a}$) parallel to $\underline{b}$: $\underline{r} = \underline{a} + \lambda \underline{b}$

Passes through C and D (position vectors $\underline{c}$ and $\underline{d}$): $\underline{r} = \underline{c} + \lambda(\underline{d} - \underline{c})$

If $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$,

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 + \lambda b_1 \\ a_2 + \lambda b_2 \\ a_3 + \lambda b_3 \end{pmatrix} \Leftarrow \underline{r} = \underline{a} + \lambda \underline{b}$ can be given in cartesian form as: $\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$ (=

Each of these three expressions is equal to λ

Vector equation of a plane: $\underline{r} = \underline{a} + \lambda \underline{b} + \mu \underline{c}$ where:

- $\underline{r}$ = the position vector of a general point on the plane
- $\underline{a}$ = position vector of a point on the plane
- $\underline{b}$ and $\underline{c}$ are non-parallel, non-zero vectors in the plane
- λ and μ are scalars

$\boxed{\underline{u} \times \underline{v} = |\underline{u}||\underline{v}|\sin(\theta)\hat{n} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}}$

If $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector to a plane, then $ax + by + cz = d$ is the cartesian equation of the plane (derived using scalar product)

Scalar product ('dot product'): $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta$ $\begin{cases} 0 \text{ if } \underline{a}, \underline{b} \text{ are } \perp \\ |\underline{a}||\underline{b}| \text{ if } \underline{a}, \underline{b} \text{ are } \parallel \end{cases}$

θ = angle between $\underline{a}$ and $\underline{b}$ when they are placed tail to tail

Should really be $\hat{i}$, $\hat{j}$ and $\hat{k}$

Hence, $\underline{i} \cdot \underline{j} = \underline{i} \cdot \underline{k} = \underline{j} \cdot \underline{i} = \underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{i} = \underline{k} \cdot \underline{j} = 0$

$\underline{a} \cdot \underline{a} = |\underline{a}|^2$

$\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$

$\cos\theta = \left| \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} \right|$ = acute angle θ between two intersecting lines

If $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$: $\underline{a} \cdot \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$ ← use this to prove this

To find the normal to $\underline{r} = \underline{a} + \lambda \underline{b} + \mu \underline{c}$, find two vectors in the plane, using ~~[illegible]~~ λ / μ. Let these be $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ and $\begin{pmatrix} g \\ h \\ i \end{pmatrix}$. This gives 2 equations in 3 unknowns, however, since normals are all multiples of each other, let $n_3 = 1$ and solve for n_1 and n_2. Need $\begin{pmatrix} d \\ e \\ f \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$ and $\begin{pmatrix} g \\ h \\ i \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$

Scalar product form of the equation of a plane is: $\underline{r} \cdot \underline{n} = k$

where $k = \underline{a} \cdot \underline{n}$ for any point in the plane with position vector $\underline{a}$

Cartesian form: $n_1 x + n_2 y + n_3 z - k = 0$

Or, as in formula book: $n_1 x + n_2 y + n_3 z + d = 0$, where $d = -\underline{a} \cdot \underline{n}$

Vectors cont.

Angle between line and plane: $\sin\theta = \left| \dfrac{\underline{b} \cdot \underline{n}}{|\underline{b}||\underline{n}|} \right|$

Angle between two planes: $\cos\theta = \left| \dfrac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1||\underline{n}_2|} \right|$

Intersection of a line and a plane:

line: $\underline{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ $\Rightarrow$ plane: $\underline{r} \cdot \underline{n} = K$ $\Rightarrow$ solve for λ and sub into

$\underline{r} = \begin{pmatrix} a_1 + \lambda d_1 \\ a_2 + \lambda d_2 \\ a_3 + \lambda d_3 \end{pmatrix}$ $\quad \begin{pmatrix} a_1 + \lambda d_1 \\ a_2 + \lambda d_2 \\ a_3 + \lambda d_3 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = K$ $\quad \begin{pmatrix} a_1 + \lambda d_1 \\ a_2 + \lambda d_2 \\ a_3 + \lambda d_3 \end{pmatrix} = \underline{r}$

Intersection of two lines: $L_1: \underline{r} = \underline{a} + s\underline{b}$ $\quad L_2: \underline{r} = \underline{c} + t\underline{d}$

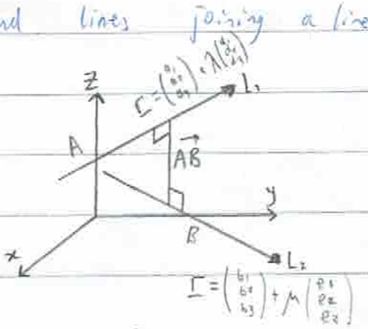
↳ Test to see if parallel and to see if they intersect

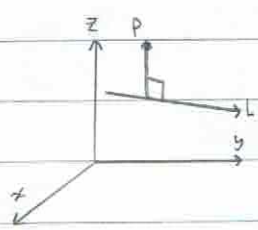
Is $\underline{d}$ a multiple of $\underline{b}$?

- No → Equate x, y, z components to find 3 equations in s & t
  - First two equations' solutions satisfy the 3rd → Unique solution: intersect
  - No solution → No solution: skew
- Yes → Equate x, y, z components to find 3 equations in s & t
  - Equations have a contradiction → No solution: parallel
  - Equations are the same → Infinite solutions: coincident

Finding Perpendicular distances

Basically, to do this just draw out a diagram, figure out what is perpendicular to what, and then do 1st thing · 2nd thing = 0. This will very likely involve finding general equations for lines, lines joining two lines, and lines joining a line to a point.





Useful to know/do: A = general point on L_1 = $\begin{pmatrix} a_1 + \lambda d_1 \\ a_2 + \lambda d_2 \\ a_3 + \lambda d_3 \end{pmatrix}$

B = general point on L_2 = $\begin{pmatrix} b_1 + \mu e_1 \\ b_2 + \mu e_2 \\ b_3 + \mu e_3 \end{pmatrix}$

remember to do $|\vec{AB}|$ at end

Find a general equation for $\vec{AB}$ ($\vec{AB} = B - A$) (if parallel lines ~~use t=μ~~)

K is the length of the perpendicular from the origin to a plane Π, where the plane equation is $\underline{r} \cdot \hat{\underline{n}} = K$ where $\hat{n}$ is a unit vector perpendicular to Π.

The perpendicular distance from the point with coordinates (α, β, γ) and the plane with equation $ax + by + cz = d$ is: $\left| \dfrac{a\alpha + b\beta + c\gamma - d}{\sqrt{a^2 + b^2 + c^2}} \right|$

Vectors cont.

Reflections of points and lines in planes

Points :
- ~~Find shortest distance from P to plane~~
- Find vector equation of ~~this line,~~ the shortest line from p to π (it's in the direction of normal and passes through P)
- Find the point of intersection of the line and plane by substituting in $\underline{r}$ from line into $\underline{r} \cdot \underline{n} = k$ to find λ. (let this point be Q)
- Reflection of P has position vector $\underline{p}' = \underline{p} + 2(\underline{q} - \underline{p})$

lines
- Reflect 2 points in the line
- Make one of the points P the intersection of the line and the plane so you don't have to reflect it
- Make the other point Q by arbitrarily setting λ to 0
- Now find the equation of the line passing through P and Q

Further Series

If the general term, u_r, of a series can be represented as $f(r) - f(r+1)$

then $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} (f(r) - f(r+1))$

$$u_1 = f(1) - \cancel{f(2)}$$
$$u_2 = \cancel{f(2)} - \cancel{f(3)}$$
$$\dots$$
$$u_{n-1} = \cancel{f(n-1)} - \cancel{f(n)}$$
$$u_n = \cancel{f(n)} - f(n+1)$$

$$\Rightarrow \sum_{r=1}^{n} u_r = f(1) - f(n+1)$$

Maclaurin series : $f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x}{1!} + \frac{f^2(0)x^2}{2!} + \dots + \frac{f^n(0)x^n}{n!}$

$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots$ for all x

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1}\frac{x^r}{r} + \dots$ $-1 < x \leq 1$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots$ for all x

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots$ for all x

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^r \frac{x^{2r+1}}{2r+1} + \dots$ $-1 \leq x \leq 1$

Methods in calculus

The integral $\int_a^b f(x)\,dx$ is improper if either:

- one or both of the limits is infinite
- $f(x)$ is undefined at $x=a$, $x=b$ or at another point in the interval $[a,$

Mean value of $f(x)$ over the interval $[a,b]$: mean $= \frac{1}{b-a}\int_a^b f(x)\,dx$ (= area / width)

If $f(x)$ has a mean value $\bar{f}$ over the interval $[a,b]$, and k is a real constant, then:

- $f(x) + k$ has mean value $\bar{f} + k$
- $k f(x)$ has mean value $k\bar{f}$
- $-f(x)$ has mean value $-\bar{f}$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c, \quad a > 0, \quad |x| < a$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + c$$

Polar Coordinates

$r\cos\theta = x$ and $r\sin\theta = y$

$r^2 = x^2 + y^2$ $\qquad \theta = \arctan\left(\frac{y}{x}\right)$ ← ensure in the correct quadrant

circle centre O radius a: $r = a$

half-line through O with angle α: $\theta = \alpha$

spiral starting at O: $r = a\theta$

Area bounded by polar curve, $\theta = \alpha$, and $\theta = \beta$: Area $= \frac{1}{2}\int_\alpha^\beta r^2 \, d\theta$

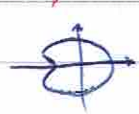
To find a tangent parallel to the initial line: $\frac{dy}{d\theta} = 0$

To find a tangent perpendicular to the initial line: $\frac{dx}{d\theta} = 0$

Cardioids

Curves with equations of the form $r = a(p + q\cos\theta)$ are defined for all θ if $p \geq q$

The curve is convex (egg shaped) if $p \geq 2q$ and concave if $2q > p \geq q$

convex =  concave = 

Differential Equations

First Order

You can solve first-order differential equations of the form $\frac{dy}{dx} + P(x)y = Q(x)$ by multiplying every term by the integrating factor (IF) $e^{\int P(x)\,dx}$.

Second Order Homogeneous

The second order homogeneous differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ can be written as an auxillary equation (AU) $am^2 + bm + c = 0$. The roots of AU determine the general solution (GS) of the differential equation

Case 1:	2 distinct real roots α and β GS ⇒ $y = Ae^{\alpha x} + Be^{\beta x}$
Case 2:	1 repeated root α GS ⇒ $y = (A + Bx)e^{\alpha x}$
Case 3:	2 complex conjugate roots $p \pm qi$ GS ⇒ $y = e^{px}(A\cos qx + B\sin qx)$

Second Order Non-homogeneous

Equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ are non-homogeneous.
First you need to find the general solution of the corresponding homogeneous differential equation, $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$. This is called the complimentary function (CF).
Next you need to find a particular integral (PI). This is a function that satisfies the differential equation. PI depends on $f(x)$.

Forms of $f(x)$	Form of PI
p	λ
$p + qx$	$\lambda + \mu x$
$p + qx + rx^2$	$\lambda + \mu x + \nu x^2$
pe^{kx}	λe^{kx}
$p\cos\omega x + q\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

if this form can be ~~found~~ already found in the CF, you need to multiply by x or x^2.

To then find the GS: $y = CF + PI$

Using Differential Equations to Model:

Simple Harmonic Motion

S.H.M: Motion in which the acceleration of a particle P is always towards a fixed point O on the line of motion of P. The acceleration is proportional to the displacement of P from O.

$\ddot{x} = -\omega^2 x$ $\qquad \ddot{x} = v\frac{dv}{dx}$ $\qquad \dot{x} = \frac{dx}{dt} = v$

$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = a$

Damped Harmonic Motion

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt} - \omega^2 x$$

if you need to find angular velocity, you can't use this constant, instead if $x = f(x)(R\sin(kt+\alpha))$ $\omega = k$

proportional to 'damping force'

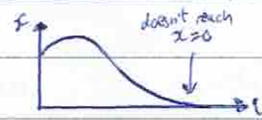
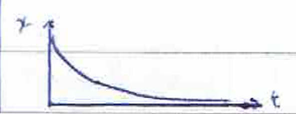
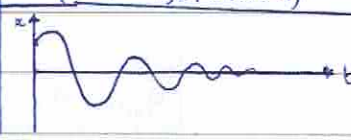
proportional to 'restoring force'

or $\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$$

⇩

AU: $m^2 + km + \omega^2 = 0$

Roots of auxiliary	Distinct Roots $k^2 - 4\omega^2 > 0$	Equal Roots $k^2 - 4\omega^2 = 0$	No Roots $k^2 - 4\omega^2 < 0$
Form of resulting solution	$x = Ae^{\alpha t} + Be^{\beta t}$	$x = (A+Bt)e^{\alpha t}$	$x = Ae^{\alpha t}\sin(\beta x + \gamma)$
Type of damping	Heavy damping (no oscillations)	Critical damping (the limit for which there are no oscillations)	Light damping (some oscillations)
Sketch of x against t			

Forced Harmonic Motion

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = f(t)$$

To get equations in this form just use $\Sigma F = ma$

Coupled First-Order Linear Differential Equations

$$\frac{dx}{dt} = ax + by + f(t)$$

$$\frac{dy}{dt} = cx + dy + g(t)$$

Homogeneous if $f(t) = g(t) = 0$ for all t

Strategy to Solve for x:

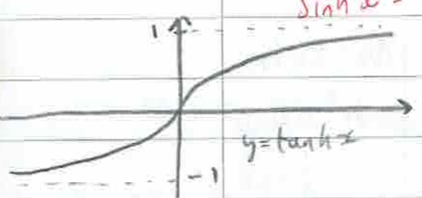
1. Make y the subject of the equation for $\frac{dx}{dt}$ then differentiate to find $\frac{dy}{dt}$

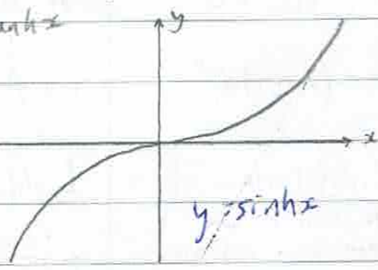
May have lots of As and Bs, This is okay.

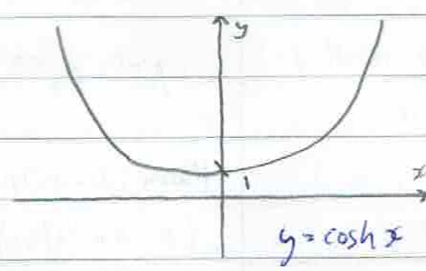
2. Substitute y and $\frac{dy}{dt}$ into the second equation to get a single second-order differential equation just in terms of x, and solve to get an equation for x.
3. To solve for y, no need to restart. Just differentiate your new equation for x and then substitute ~~x~~ x and $\frac{dx}{dt}$ into y from step 1

Hyperbolic Functions

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ $\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\tanh x = \dfrac{e^{2x} - 1}{e^{2x} + 1}$







$\sinh(-a) = -\sinh(a)$

$\cosh(-a) = \cosh(a)$

Hyperbolic function	Inverse Hyperbolic function
$y = \sinh x$	$y = \text{arsinh } x$
$y = \cosh x$	$y = \text{arcosh } x$ $x \geq 1$
$y = \tanh x$	$y = \text{artanh } x$ $\lvert x \rvert < 1$

$\text{arsinh } x = \ln\left(x + \sqrt{x^2+1}\right)$

$\text{arcosh } x = \ln\left(x + \sqrt{x^2-1}\right)$ $x \geq 1$

$\text{artanh } x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $\lvert x \rvert < 1$

$f(x) / \int g(x)\,dx$	$f'(x) / g(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\text{arsinh } x$	$1/\sqrt{x^2+1}$
$\text{arcosh } x$	$1/\sqrt{x^2-1}$
$\text{artanh } x$	$1/1-x$
$\ln \cosh x$	$\tanh x$

$\cosh^2 A - \sinh^2 A \equiv 1$

$\sinh(A \pm B) \equiv \sinh A \cosh B \pm \cosh A \sinh B$

$\cos(A \pm B) \equiv \cosh A \cosh B \pm \sinh A \sinh B$

$$\left(\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \text{arsinh}\left(\frac{x}{a}\right) + c,\quad \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \text{arcosh}\left(\frac{x}{a}\right) + c\right)$$